JOURNAL
OF THE
WASHINGTON ACADEMY OF SCIENCES

VOL. 18 OCTOBER 4, 1928 No. 16

MATHEMATICAL STATISTICS.—*On damping effects and approach to equilibrium in certain general phenomena.*[1] BIRGER MEIDELL, University of Oslo, Norway (Communicated by RAYMOND PEARL).

1. INTRODUCTION

In various branches of applied mathematics one is often confronted with problems which involve a certain function of the time, called the *resultant* $W(t)$ in what follows, that can be looked upon as built up through a cumulative process started and continually kept up by another function of time, the *originator* $n(t)$, the effect of which on $W(t)$ is brought about through the intervention of a certain function of *elapsed* time, the *distributor* $w(\xi)$.

The following are some instances drawn from the field of population statistics: The total existing population, or certain subgroups of the population, such as the number of men, or women, or men in certain age groups, at a certain moment of time, or the annual number of deaths or of marriages, are all resultants, i.e., functions of time which are originated through another function of time, namely the number of births in previous years; in these examples, the distributors are certain functions derived from the age distributions at death, at marriage, etc. Again, the total reserve at a given moment for a particular kind of insurance is a resultant which is originated through the "production," i.e., the new insurance written each year; here the distributor is the

[1] Received June 8, 1928. The present paper is a revised and somewhat generalized version of the author's article *Ueber periodische und angenäherte Beharrungszustände*, Skandinavisk Aktuarie Tidskrift. 1926.

function of elapsed time that represents the total reserve resulting from a unit of production of one year for the kind of insurance considered.[2]

In all the above cases the effect of the distributor is *limited* by the maximum length of human life. An analogous limitation is present in a great variety of cases occurring in other fields.

The most important case is perhaps that in which the cumulative process is such that the originator-element $n(\tau)d\tau$ for the moment of time $t = \tau$ is the cause of a contribution of amount $w(\xi)n(\tau)d\tau$ to the resultant at the subsequent moment of time $t = \tau + \xi$. We then have, since $w(\xi) = w(t - \tau)$,

$$W(t) = \int_{-\infty}^{t} w(t - \tau)\, n(\tau)\, d\tau \ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

As, however, the originating process will in general start at some definite time which we may select for $t = 0$, we may assume $n(\tau) = 0$ for $\tau < 0$; thus we obtain

$$W(t) = \int_{0}^{t} w(t - \tau)\, n(\tau)\, d\tau \ldots\ldots\ldots\ldots\ldots\ldots (1.2)$$

If the originator $n(t)$ is a constant n, we have

$$W(t) = n \int_{0}^{t} w(t - \tau)\, d\tau \ldots\ldots\ldots\ldots\ldots\ldots (1.3)$$

which, upon substituting $t - \tau = \xi$, $d\tau = -d\xi$ (since t is a definite fixed time), may also be written

$$W(t) = n \int_{0}^{t} w(\xi)\, d\xi \ldots\ldots\ldots\ldots\ldots\ldots (1.4)$$

If the effect of the distributor is limited, i.e., $w(\xi) = 0$ for $\xi \geq T$, where T is some positive constant, we get for values of $t \geq T$

$$W(t) = n \int_{0}^{T} w(\xi)\, d\xi = const. \ldots\ldots\ldots\ldots\ldots\ldots (1.5)$$

This last equation expresses a well known equilibrium theorem.

[2] The advantages of considering actuarial reserves from this point of view has been pointed out by the writer in an article in Skandinavisk Aktuarie Tidskrift, 1921, p. 210. See also the work of Norwegian actuaries in Nyt beregningsgrundlag for livsforsikring, Oslo, 1922, p. 27, and Fr. Lange-Nielsen, *A proposal for a new basis . . . .*, Skand. Akt. T., 1922, p. 235.

In practical cases where the originator and the distributor are known with sufficient accuracy, these equations give a means of forecasting the magnitude of the resultant. In particular, if n = *const.*, and the distributor is limited, equation (1.5) gives a means of predicting the time from which there will be equilibrium in the sense of W = *const.* It is obvious that it would be of great practical importance to obtain criteria for cases in which equilibrium will be approximately reached when the originator $n(t)$ is not constant; the attempt to do this is the chief aim of the present paper.

2. The Originator and the Distributor as Solutions of Integral Equations

In the following we shall write equation (1.2) in the form

$$W(t) = \int_0^t w(\xi)\, n(t-\xi)\, d\xi = \int_0^t w(t-\tau)\, n(\tau)\, d\tau \ldots (2.1)$$

A generalization of this equation would be

$$W(t) = \int_0^t \sum_i w_i(\xi)\, n_i(t-\xi)\, d\xi \ldots\ldots\ldots\ldots (2.2)$$

where w_i and n_i are two series of distributors and originators; if either the w_i or the n_i are identically proportional, this case reduces to (2.1). In the present paper, attention will be confined to the case (2.1).

Equations (2.1) are *integral* equations, in $w(\xi)$ and in $n(\tau)$ respectively, of the so-called generalized Abel type; to obtain practical solutions of them, it is desirable (when the nucleus remains finite) to transform them by differentiation with respect to t, whence we obtain[3]

$$\frac{W'(t)}{n(0)} = w(t) + \frac{\int_0^t w(\xi)\, n'(t-\xi)\, d\xi}{n(0)} \ldots\ldots\ldots (2.3)$$

$$\frac{W'(t)}{w(0)} = n(t) + \frac{\int_0^t n(\tau)\, w'(t-\tau)\, d\tau}{w(0)} \ldots\ldots\ldots (2.4)$$

If in (2.3) the quantity $n(0)$ vanishes, the differentiation must be repeated until $n^{\nu}(0) \neq 0$; and similarly if $w(0)$ vanishes in (2.4). Equations (2.3) and (2.4) are integral equations, in w and n, respec-

[3] E. B. Wilson, *Advanced Calculus*, p. 283.

tively, of the so-called Poisson type; the existence theorems established by Volterra[4] give a solution of the form

$$w(t) = \frac{W'(t)}{n(0)} - \frac{\int_0^t \Gamma(t-\xi)\, W'(\xi)\, d\xi}{n(0)} \quad \ldots\ldots\ldots (2.5)$$

or, transforming the variable of integration from ξ to $\tau = t - \xi$, $d\tau = -d\xi$,

$$w(t) = \frac{W'(t)}{n(0)} - \frac{\int_0^t \Gamma(\xi)\, W'(t-\xi)\, d\xi}{n(0)} \quad \ldots\ldots\ldots (2.6)$$

in which $\Gamma(t-\xi)$ is called the resolvent kernel or solving function; numerical methods for determining Γ have been given by Whittaker,[5] Kameda,[6] and others. If the kernel $n'(t-\xi)/n(0) \equiv K(t-\xi)$ in (2.3) is of the form $K(x) = \Sigma_i\, a_i \exp(c_i x)$, where the a_i and c_i are constants, Γ will be of the same form $\Gamma(x) = \Sigma_i\ \alpha_i \exp(\gamma_i x)$, where the α_i and γ_i are simple functions of the a_i and c_i. In particular, if $n(t) = \alpha + \beta \sin \lambda t$, $K(t-\xi) = 2b \cos \lambda (t-\xi)$, where α, β, λ, b, are constants, we have

$$\left.\begin{aligned} \Gamma(t) &= 2be^{-bt}(\cos \mu t - \nu \sin \mu t),\ \lambda^2 > b^2 \\ \Gamma(t) &= be^{-bt}[(1-\nu)\, e^{\mu t} + (1+\nu)\, e^{-\mu t}],\ \lambda^2 < b^2 \\ \mu &= b/\nu = +\sqrt{|\lambda^2 - b^2|} \end{aligned}\right\} \quad \ldots\ldots (2.7)$$

3. Some General Relations between the Fluctuations of the Originator and the Value of the Resultant at a given Instant

Let $W_0(t)$ be the function which expresses what the resultant would have been at time t (with the given form of w) if the originator had been equal to a constant n_0; and let us find an expression for the *deviation* $U(t) = W(t) - W_0(t)$.

If $n_0 = n(0)$, we have from (1.4) and (2.6), remembering that $W(0) = 0$ and that the value of a definite integral is independent of the particular variable in terms of which the integrand is expressed,

[4] Goursat, *Cours d'Analyse*, 3 ed., T. III, chap. xxx.

[5] Whittaker. Proc. Roy. Soc. Lond. **94**: 367. 1918. Whittaker and Robinson, *The Calculus of Observations*, p. 376.

[6] Kameda, Tôhoku Math. Journ. August, 1924.

$$U(t) = W(t) - \int_0^t n(0)\, w(t)\, dt$$

$$= \int_0^t d\tau \int_0^\tau \Gamma(\xi)\, W'(\tau - \xi)\, d\xi$$

or, interchanging the order of the integrations by Dirichlet's Formula,[7]

$$U(t) = \int_0^t d\xi \int_\xi^t \Gamma(\xi)\, W'(\tau - \xi)\, d\tau$$

Hence, evaluating the second integral (in which W' is now the derivative with respect to τ), we have

$$U(t) = \int_0^t \Gamma(\xi)\, W(t - \xi)\, d\xi = \int_0^t \Gamma(t - \xi)\, W(\xi)\, d\xi$$

Again, from (2.1) we get directly

$$U(t) = \int_0^t w(\xi)\, n(t - \xi)\, d\xi - n_0 \int_0^t w(\xi)\, d\xi$$

The deviation $U(t)$ may be computed from either of these equations for any value of t. The last equation is also valid if n is an arbitrary constant.

Generally, let $W(t)$ and $H(t)$ be two resultants, generated by the originators $n(t)$ and $h(t)$ respectively, and with the same distributor $w(\xi)$. The function $h(t)$ will be called the trend function. The difference $u(t) = n(t) - h(t)$ will represent the deviations or fluctuations of the originator $n(t)$ measured from the trend function, and $U(t) = W(t) - H(t)$ will represent the corresponding deviations of the resultant.

We shall first suppose $h(t)$ to be an arbitrary function without any connection with $n(t)$. By (2.1)

$$\frac{U(t)}{H(t)} = \frac{\int_0^t u(t - \xi)\, w(\xi)\, d\xi}{\int_0^t h(t - \xi)\, w(\xi)\, d\xi}$$

$$= \frac{\int_0^t \frac{u(t - \xi)}{h(t - \xi)} \cdot h(t - \xi)\, w(\xi)\, d\xi}{\int_0^t h(t - \xi)\, w(\xi)\, d\xi}$$

[7] GOURSAT, *Cours d'Analyse*, 4 ed., T. I., p. 299.

Thus, by a well known property of definite integrals,

$$\left|\frac{U(t)}{H(t)}\right| \leq \frac{\int_0^t \left|\frac{u(t-\xi)}{h(t-\xi)}\right| \left|h(t-\xi)\, w(\xi)\right| d\xi}{\left|\int_0^t h(t-\xi)\, w(\xi)\, d\xi\right|}$$

Hence

$$\left|\frac{U(t)}{H(t)}\right| \leq M \frac{\int_0^t \left|h(t-\xi)\, w(\xi)\right| d\xi}{\left|\int_0^t h(t-\xi)\, w(\xi)\, d\xi\right|}$$

where M is the maximum of $|\, u(\tau)\, /h(\tau)\,|$ in the interval $0 \leq \tau \leq t$. If $h(t-\xi)\, w(\xi)$ has the same sign throughout the interval $0 \leq \xi \leq t$. which in practice it usually will have,[8] we have simply

$$\left|\frac{U(t)}{H(t)}\right| \leq M$$

Therefore we have

THEOREM I. Whatever the trend function $h(\tau)$, if the product $h(\tau)\, w(t-\tau)$ does not change sign in the interval $0 \leq \tau \leq t$, the relative deviation of the resultant $W(t)$ from the value which this function would have assumed if the originator had been the trend function $h(\tau)$ has at any given point t an absolute value not greater than the maximum absolute value of the relative deviation of the originator from the trend function in the interval $0 \leq \tau \leq t$.

In other words, since this theorem holds for any t, a damping of the relative fluctuations of $W(t)$ as compared with the relative fluctuations of $n(t)$ must in all such cases take place. Upon the intensity of this damping will depend the possibility in practice of computing $W(t)$ *as if* $n(t)$ were a suitable trend function $h(t)$ of a less complicated form than $n(t)$ but still representing the main features of $n(t)$—e.g., as if $n(t)$ were a constant in cases where the underlying trend of $n(t)$ is recognized to have the form of a horizontal line.

The damping effect will be more fully examined below, through the introduction of the standard deviation as a measure of the fluctuations of $W(t)$.

[8] In most cases, both $h(t)$ and $u(t)$ will be positive.

If the originator $n\ (t)$ has a marked variation confined to a short interval in the neighbórhood of time t, and if the distributor $w\ (x)$ is positive and has a pronounced maximum at $x = a$, then there will obviously be a more or less pronounced fluctuation in the resultant in the neighborhood of time $t + a$. We may obtain some information about this effect by considering the originator to be increased over a small finite interval of length $p = 2q$, say: Let $n_1\ (x)$ be a function which is positive in the interval $0 \leq x \leq p$, and equal to zero outside this interval; $n_1\ (x)$ might be called the "extra originator." We wish to know the effect on the resultant W when, starting from any time t, the extra originator is added to or subtracted from n.

The resultant which would be produced at any instant of time $t + y \geq t + p$ by n_1 alone, viz.,

$$W_1\ (y) = \int_0^p w\ (y - \xi)\ n_1\ (\xi)\ d\,\xi \ldots\ldots\ldots\ldots (3.1)$$

might be called the "extra resultant." The problem is to determine the maximum of this function. If there is a maximum at $y = \eta + q$, η must satisfy

$$\int_{-q}^{+q} w'\ (\eta - \xi)\ n_1\ (q + \xi)\ d\,\xi = 0$$

if $w\ (x)$ is symmetric about its maximum $x = a$, and $n_1\ (x)$ symmetric about the midpoint $x = q$, this equation will be satisfied for $\eta = a$. Hence, the maximum effect on W of a temporary, symmetric fluctuation in the originator n is produced after a lapse of time which, counted from the midpoint of the fluctuation in n, is equal to a, where a is the argument for which the symmetric distributor $w\ (x)$ has its maximum.

The conditions under which this solution holds are often realized in practise, because several distributors encountered in statistical and actuarial work, e.g., are approximately of the symmetric type. A striking instance is the reserve

$$w_t = c\left(1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}\right)\frac{l_{x+t}}{l_x}$$

of a sufficiently great number of whole life insurances entered at age x.

4. The Damping Coefficient

A formula will now be established that expresses the way in which the general character of the fluctuations of $W\ (t)$ in $z - q \leq t \leq z + q$

depends on the fluctuations of $n\ (t)$ in the time previous to $z + q$. The distributor will be assumed limited in its effect, i.e., $w\ (x) = 0$ for $x \geq T$; hence for $t \geq T$

$$W\ (t) = \int_0^T w\ (\xi)\ n\ (t - \xi)\ d\ \xi = \int_0^T w\ (t - \xi)\ n\ (\xi)\ d\ \xi \ldots (4.1)$$

Assume also that $z \geq T + q$; T is the instant of time at which equilibrium would have been reached, i.e., the time from which W would have been constant, if the originator n had been independent of time, and it will be called the *equilibrium point* regardless of whether or not equilibrium actually is reached there. The equilibrium point evidently is independent of the constant value assumed for n, and depends only on the distributor $w\ (x)$.

The function

$$\omega\ (x) = \frac{w\ (x)}{\int_0^T w\ (\xi)\ d\ \xi}$$

will be called the *normalized distributor*, or the *distribution function*. We further introduce the means

$$h\ (z) = \frac{1}{p}\int_{z-q}^{z+q} n\ (\tau)\ d\ \tau; \quad H\ (z) = \frac{1}{p}\int_{z-q}^{z+q} W\ (\tau)\ d\ \tau$$

in which $p = 2q$; if z is varying, h and H can be regarded as the moving averages of n and W.[9] The deviations of n and W from their means will be denoted by

$$u\ (t, z) = n\ (t) - h\ (z)$$

$$U\ (t, z) = W\ (t) - H\ (z)$$

It should be noted that the position of the interval which determines the means from which u and U are measured is fixed independently through z, and does not change with t.

[9] In my original analysis of the character of the oscillations of W (t) over a given interval, I considered only the case of a periodic originator, which evidently amounted to considering the trend function h to be a constant. The more general definition of h and H, above, as moving averages has been suggested to be by my friend and colleague Dr. Frisch.

We evidently have for any z

$$\int_{z-q}^{z+q} u(\tau, z)\, d\tau = \int_{z-q}^{z+q} U(\tau, z)\, d\tau = 0$$

The means and the deviations satisfy the same equations as do the functions themselves; in fact, we have by (4.1)

$$H(z) = \int_0^T w(\xi)\, h(z-\xi)\, d\xi \ldots\ldots\ldots\ldots (4.2)$$

and

$$U(t, z) = \int_0^T w(\xi)\, u(t-\xi, z-\xi)\, d\xi \ldots\ldots\ldots (4.3)$$

for $t \geq T$ and $z \geq T + q$.

The standard deviations $s(z)$ and $S(z)$ of $n(t)$ and $W(t)$ over the interval $z - q \leq t \leq z + q$ are defined by

$$s^2(z) = \frac{1}{p}\int_{z-q}^{z+q} u^2(\tau, z)\, d\tau$$

$$S^2(z) = \frac{1}{p}\int_{z-q}^{z+q} U^2(\tau, z)\, d\tau$$

Finally, we introduce the coefficient of correlation $C(2\delta, z)$ between the 2δ-spaced ordinates $n(t-\delta)$ and $n(t+\delta)$ of the originator, the correlation being taken over the interval $z - q \leq t \leq z + q$:

$$C(2\delta, z) = \frac{\frac{1}{p}\int_{z-q}^{z+q} u(\tau+\delta, z+\delta)\, u(\tau-\delta, z-\delta)\, d\tau}{s(z+\delta)\, s(z-\delta)} \ldots (4.4)$$

Since C is a correlation coefficient, we always have $-1 \leq C \leq 1$. Further $C(0, z) = 1$ for any z, which simply means that the correlation is perfect between infinitesimally spaced ordinates. C is symmetric in δ.

The relative standard deviation $r(z) = s(z)/h(z)$ measures the closeness with which the originator $n(t)$ fluctuates about its mean in the interval $z \pm q$; and the ratio $R(z) = S(z) / H(z)$ has the same significance for the resultant $W(t)$. The ratio $R(z)$ may therefore be taken as a measure of the degree to which equilibrium is approximately realized over the interval $z \pm q$.

Making use of (4.3), we have for $R(z)$, after some reductions,[10] the expression

$$R^2(z) = \frac{G^2(z)}{k^2(z)} \int_0^T \int_0^T \omega(\xi)\omega(\eta)s(z-\xi)s(z-\eta)C\left(\xi-\eta, z-\frac{\xi+\eta}{2}\right)d\xi\, d\eta \ldots (4.5)$$

where

$$G(z) = \frac{\int_0^T w(\xi)\, d\xi \int_0^T h(z-\xi)\, d\xi}{T\int_0^T w(\xi)\, h(z-\xi)\, d\xi} \ldots\ldots\ldots (4.6)$$

$$k(z) = \frac{1}{T}\int_0^T h(z-\xi)\, d\xi$$

We can further reduce (4.5) by means of the following formula due to Frisch:[11] Let $f(x_1, \ldots, x_n)$ or, for brevity, f_x, be a function of the n variables $x_1, \ldots, x_n$, and let $\int_x$ denote $\int_a^b \ldots \int_a^b dx_1 \ldots dx_n$. Further let $m_f = \frac{1}{(b-a)^2}\int_x f_x$ be the mean of f, and let

[10] The reductions consist in writing

$$U^2(\tau, z) = \int_0^T d\xi \int_0^T d\eta\, w(\xi)\, w(\eta)\, u(\tau-\xi, z-\xi)\, u(\tau-\eta, z-\eta)$$

then transforming the resulting triple integral

$$\int_{z-q}^{z+q} d\tau \int_0^T d\xi \int_0^T d\eta \ldots\ldots\ldots$$

into

$$\int_0^T d\xi \int_0^T d\eta \int_{z-q}^{z+q} d\tau \ldots\ldots\ldots$$

and finally putting

$$\tau = \tau' + \frac{\xi+\eta}{2}$$

[11] RAGNAR FRISCH. *On approximation to a certain type of integrals.* Skandinavisk Aktuarie Tidskrift. 1928. The formula there given is a generalization of a formula which Frisch had previously given; for other generalizations of Frisch's formula see Steffensen. *On the sum or integral of the product of two functions.* ibid., p. 44. 1927.

$$\sigma_f = + \sqrt{\frac{1}{(b-a)^n} \int_z (f_z - m_f)},$$

$$\rho_f = \frac{\sigma_f}{m_f}$$

(m_f assumed $\neq 0$) be the standard deviation and the relative standard deviation, respectively, of f. Finally, let

$$r_{fg} = \frac{\int_z (f_z - m_f)(g_z - m_g)}{\sigma_f \sigma_g (b-a)^n}$$

be the coefficient of correlation between the two functions f_z and g_z. Then we have

$$(b-a)^n \int_z f_z g_z = \int_z f_z \cdot \int_z g_z (1 + \rho_f \rho_g r_{fg})$$

the means m_f and m_g being assumed $\neq 0$. Substituting for ρ_f, ρ_g and r_{fg}, we see that this is an identity.

Applying this to the two functions of two variables

$$f(\xi, \eta) = s(z - \xi)\, s(z - \eta)$$

$$g(\xi, \eta) = \omega(\xi)\, \omega(\eta)\, C\left(\xi - \eta, z - \frac{\xi + \eta}{2}\right)$$

in (4.5), and also to the two functions of one variable $w(\xi)$ and $h(z-\xi)$ in (4.6), we have

$$R(z) = D(z) \frac{\int_0^T s(z - \xi)\, d\xi}{\int_0^T h(z - \xi)\, d\xi} \quad \ldots\ldots\ldots\ldots (4.7)$$

where

$$D^2(z) = \frac{1 + \rho_f \rho_g r_{fg}}{(1 + \rho_w \rho_h r_{wh})^2} \int_0^T \int_0^T \omega(\xi)\, \omega(\eta)\, C\left(\xi - \eta, z - \frac{\xi + \eta}{2}\right) d\xi\, d\eta \ldots (4.8)$$

The ratio of the integrals in (4.7) may be regarded as an average relative standard deviation of the originator for the interval previous to $z + q$. We may therefore adopt the factor $D(z)$ in (4.7) and (4.8)

as the *coefficient of damping*, which expresses the ratio between the relative standard deviation of the resultant over the interval $z \pm q$ and the average relative standard deviation of the originator over the interval previous to $z + q$; the magnitude of $D(z)$ measures the intensity of the damping. If $D(z)$ is a small fraction, a heavy damping has taken place. It is true that a small value of $D(z)$ does not exclude the possibility that there may have taken place, within the interval $z \pm q$, *isolated* deviations of W of comparatively great magnitude, but such fluctuations must necessarily have been confined to very short intervals of time; in practise, such extreme cases will be of little importance.

The expression (4.8) enables us to recognize certain cases in which $D(z)$ will be small; we shall consider, in particular, the case of a periodic originator.

5. The Case of a Periodic Originator

If the moving average h is constant, i.e., if the originator is periodic with period p, then h, s, H, and S (for $z \geq T + q$) will be constants, hence $\sigma_f = \sigma_h = 0$. Furthermore, $u(t, z)$ and therefore $C(\delta, z)$ will be independent of z. The ratios r and R, and hence also D, will be constants. We have merely $R = Dr = Ds/h$, and

$$D^2 = \int_0^T \int_0^T \omega(\xi)\, \omega(\eta)\, C(\xi - \eta)\, d\xi\, d\eta \ldots\ldots\ldots (5.1)$$

Since $w(x) = 0$ for $x \geq T$, we have further

$$W(t + p) - W(t) = \int_0^T w(\xi)\, [n(t + p - \xi) - n(t - \xi)]\, d\xi = 0$$

for $t \geq T$. Hence we have

Theorem II. If the originator is an arbitrary periodic function, the resultant will be periodic after equilibrium has been reached, and the length of its period will be equal to the length of the period of the originator.

This condition, realized from the point of time T on, might be termed periodic equilibrium. For the constants h and H we have

$$h = \frac{1}{p} \int_c^{c+p} n(\tau)\, d\tau, \quad H = \frac{1}{p} \int_c^{c+p} W(\tau)\, d\tau$$

where $c \geq T$ but otherwise arbitrary. Furthermore, from (4.2)

$$H = h \int_0^T w(\xi)\, d\xi$$

whence

THEOREM III. If the originator is periodic, the actual mean value of the resultant (after equilibrium) is equal to the constant value which the resultant would have had (after equilibrium) if the originator had been constant and equal to its actual mean value.

For the function $C(\delta) = C(\delta, z)$, now independent of z, we have

$$\alpha\, C(\delta) = \int_c^{c+p} u(\tau + \delta)\, u(\tau)\, d\tau \ldots\ldots\ldots\ldots (5.2)$$

where

$$\alpha = \int_c^{c+p} u^2(\tau)\, d\tau$$

Since $\alpha = 0$ is a trivial case, we may assume $\alpha > 0$; $C(\delta)$ is evidently independent of the constant c in (5.2), and by integrating (5.2) over δ we find

$$\int_c^{c+p} C(\delta)\, d\delta = 0$$

Furthermore, $C(\delta)$ is an even periodic function with period p: $C(\delta + p) = C(\delta)$, $C(\delta) = C(-\delta)$. Hence C is symmetric about $\delta = kp$, where k is an arbitrary positive or negative integer or zero; C is even symmetric about $\delta = kq$, where $p = 2q$, for $\alpha\, C(q + \delta) = \alpha\, C(-q - \delta)$ is equal to

$$\int_c^{c+p} u(\tau - q - \delta)\, u(\tau)\, d\tau = \int_c^{c+p} u(\tau + q - \delta)\, u(\tau)\, d\tau = \alpha\, C(q - \delta)$$

If u possesses derivatives of all orders, then (5.2) gives by ν repeated integrations by parts,

$$\alpha\, C^{(m)}(\delta) = (-1)^\nu \int_c^{c+p} u^{(m-\nu)}(\tau + \delta)\, u^{(\nu)}(\tau)\, d\tau \ldots\ldots (5.3)$$

From this, we have for $m = 2\rho + 1$, $\nu = \rho$ $(\rho = 0, 1, \ldots)$,

$$2\alpha\, C^{(2\rho+1)}(0) = (-1)^\rho \int_c^{c+p} d\,[u^{(\rho)}(\tau)]^2 = 0$$

because $[u^{(\rho)}(\tau)]^2$ is periodic with period p. Hence $C^{(2\rho+1)}(0) = 0$, since $\alpha > 0$. For $m = 2\rho$, $\nu = \rho$, we get $C^{(2\rho)}(0) = (-1)^\rho \alpha_{2\rho}$, where

$$\alpha_{2\rho} = \frac{1}{\alpha}\int_c^{c+p} [u^{(\rho)}(\tau)]^2 \, d\tau$$

We thus obtain the Taylor series (assumed convergent)

$$C(\delta) = \sum_{\rho=0}^{\infty} (-1)^\rho \frac{\alpha_{2\rho}}{(2\rho)!}\delta^{2\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5.4)$$

Putting $m = 1, \nu = 0$, and $m = 1, \nu = 1$, in (5.3), and adding, we have

$$2\alpha C'(\delta) = \int_c^{c+p} [u'(\tau + \delta) - u'(\tau - \delta)]\, u(\tau)\, d\tau$$

Hence $C'(q) = 0$.

We shall next develop C in a Fourier series. Putting

$$\delta = \frac{px}{2\pi} \qquad C(\delta) = \dot{C}(x)$$

$$t = \frac{py}{2\pi} \qquad u(t) = \dot{u}(y)$$

we have

$$\dot{C}(x) = \dot{C}(-x), \int_c^{c+2\pi} \dot{C}(x)\, dx = 0, \int_c^{c+2\pi} \dot{u}(y)\, dy = 0$$

In the expansion

$$\dot{C}(x) = \sum_{m=1}^{\infty} a_m \cos mx$$

assumed convergent, the coefficients a_m are given by

$$\pi a_m = \int_c^{c+2\pi} \dot{C}(x) \cos mx \, dx$$

Introducing the expression for $\dot{C}$ in terms of $\dot{u}$, derived from (5.2), we get

$$\dot{\alpha}\pi a_m = \int_c^{c+2\pi}\int_c^{c+2\pi} \dot{u}(x+y)\,\dot{u}(y)\cos mx\, dx\, dy \ldots (5.5)$$

where

$$\dot{\alpha} = \int_c^{c+2\pi} \dot{u}^2 (y)\, dy$$

Assuming that $\dot{u}(y)$ can be developed in a Fourier series, let β_μ and γ_μ be the coefficients of this development, so that

$$\dot{u}(y) = n\left(\frac{py}{2\pi}\right) - \frac{1}{p}\int_c^{c+p} n(\tau)\, d\tau$$

$$= \sum_{\mu=1}^{\infty} (\beta_\mu \cos \mu y + \gamma_\mu \sin \mu y, \ldots\ldots\ldots\ldots (5.6)$$

Now substitute for $\dot{u}(x + y)$ in (5.4), leaving $\dot{u}(y)$ as it stands; expanding the terms $\cos(\mu x + \mu y)$ and $\sin(\mu x + \mu y)$, we see that in the integration with respect to x all terms containing $\sin \mu x . \cos mx$, as well as all terms containing $\cos \mu x . \cos mx$, $m \neq \mu$, will vanish, while

$$\int_c^{c+2\pi} \cos^2 mx\, dx = \pi$$

Introducing, next, the Fourier development of $\dot{u}(y)$ and integrating with respect to y, we get

$$\dot{\alpha}\, a_m = \pi (\beta_m^2 + \gamma_m^2)$$

From Parseval's Theorem[12], we have

$$\dot{\alpha} = \int_c^{c+2\pi} \dot{u}^2 (y)\, dy = \pi \sum_{\mu=1}^{\infty} (\beta_\mu^2 + \gamma_\mu^2)$$

Hence, finally

$$C(\delta) = \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi m \delta}{p}\right) \ldots\ldots\ldots\ldots\ldots (5.7)$$

where

$$a_m = \frac{\beta_m^2 + \gamma_m^2}{\sum_{\mu=1}^{\infty} (\beta_\mu^2 + \gamma_\mu^2)} \ldots\ldots\ldots\ldots\ldots\ldots (5.8)$$

β_μ and γ_μ being defined by (5.6). Since all the a_m are positive, it follows again that $a_1 + a_2 + \ldots = C(0) = 1$ is the greatest value

[12] WHITTAKER and WATSON. *Modern Analysis*, 4 ed., p. 182.

that $C(\delta)$ can assume. Summing up our results concerning $C(\delta)$, we have

THEOREM IV. Let $u(t)$ be an arbitrary periodic function with period $p = 2q$, and such that $\int_c^{c+p} u(\tau)\, d\tau = 0$. Then the function $C(\delta)$ defined by (5.1) is also periodic with period p, and $\int_c^{c+p} C(\delta)\, d\delta = 0$. If k is an arbitrary positive or negative integer, or zero, $C(\delta)$ is symmetric about all the points $\delta = kq$, and the first derivative $C'(\delta)$ vanishes at all these points; $C(\delta)$ attains at the points $\delta = kp$ its greatest value, viz., unity. The Taylor and the Fourier expansions of $C(\delta)$ are given by (5.4) and (5.7), provided these series are convergent.

This shows that even though the fluctuations of the periodic originator $n(t)$ be quite arbitrary, the function $C(\delta)$ which occurs in the damping coefficient is subject to conditions that put definite restrictions on the character of its variation.

It may be seen that $\cos 2\pi\, \delta/p$ is a function which satisfies the conditions to which $C(\delta)$ is subject.

6. THE DAMPING COEFFICIENT IN THE PERIODIC CASE

Substituting the Fourier expansion (5.7) in (4.8), we get

$$D^2 = \sum_{m=1}^{\infty} a_m \left\{ \left[\int_{-\infty}^{+\infty} \omega(\xi) \cos \lambda_m \xi\, d\xi \right]^2 + \left[\int_{-\infty}^{+\infty} \omega(\xi) \sin \lambda_m \xi\, d\xi \right]^2 \right\} \quad (6.1)$$

where $\lambda_m = 2\pi\, m/p$. For convenience, the integration has been extended over the interval $(-\infty, +\infty)$; this is allowable, since by definition $\omega(x) = 0$ for $x < 0$ and $x \geq T$.

We shall now introduce the expansion of $\omega(x)$ in a series of Hermite polynomials or parabolic-cylinder functions:[13] Let a and b be two arbitrary constants, and put $z = x - a$; then

$$\omega(x) = \omega(z + a) = f(z) = \sum_{\nu=0}^{\infty} \frac{k_\nu}{\nu!} \varphi_\nu(z) \quad (6.2)$$

[13] ARNE FISHER. *Mathematical Theory of Probabilities*, vol. I. The convergence of this series is discussed in a recent paper by CRAMÉR, *On some classes of series used in mathematical statistics*. Den Sjette Skandinaviske Matematikerkongres i Köbenhavn pp. 399–425. 1926. See also T. KAMEDA, *Theory of Generating Functions and its application to the theory of probability*, Journ. Faculty of Science Imp. Univ. Tokyo, Sec. 1, 1 (1): 1–62. 1925. *Proc. Tokyo Math.-Phys. Soc.*, (2) 8: 262–295, 336–360. 1915. The H_ν are tabulated by Jörgensen, *Undersögelser over frequensflader og korrelation*, Köbenhavn. 1916. *Cf.* Whittaker and Watson, *Modern Analysis*, chap. xvi.

in which

$$\varphi_0(z) = \varphi(z) = \frac{1}{b\sqrt{2\pi}} \exp\left(-\frac{z^2}{2b^2}\right)$$

$$\varphi_\nu(z) = \frac{d^\nu}{dz^\nu}\varphi(z) = (-b)^{-\nu} H_\nu\left(\frac{z}{b}\right)\varphi(z)$$

$$H_\nu(z) = (-1)^\nu e^{z^2/2} \frac{d^\nu}{dz^\nu} e^{-z^2/2}$$

$$k_\nu = (-b)^\nu \int_{-\infty}^{+\infty} f(z)\, H_\nu\left(\frac{z}{b}\right) dz \quad \ldots\ldots\ldots\ldots (6.3)$$

the H_ν being the Hermite polynomials. Substituting (6.2) in the general term of (6.1), and expanding $\cos \lambda (z + a)$ and $\sin \lambda (z + a)$, we see that it is necessary to compute the two integrals

$$\left.\begin{aligned} J_\nu &= \int_{-\infty}^{+\infty} \cos \lambda z \cdot \varphi_\nu(z)\, dz \\ K_\nu &= \int_{-\infty}^{+\infty} \sin \lambda z \cdot \varphi_\nu(z)\, dz \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (6.4)$$

in which for brevity we have written $\lambda = \lambda_m$. Since $\varphi_\nu(z) = (-1)^\nu \varphi_\nu(-z)$, we have $J_{2\rho+1} = K_{2\rho} = 0$. Integrating by parts, we have from the second of (6.4), $K_{2\rho+1} = -\lambda J_{2\rho}$. To determine $J_{2\rho}$, 2ρ integrations by parts of the first of (6.4) give

$$J_{2\rho} = \frac{(-1)^\rho \lambda^{2\rho}}{b\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cos \lambda z \cdot \exp\left(-\frac{z^2}{2b^2}\right) dz$$

$$= (-1)^\rho \lambda^{2\rho} \exp\left[-\frac{(\lambda b)^2}{2}\right]$$

hence

$$K_{2\rho+1} = (-1)^{\rho+1} \lambda^{2\rho+1} \exp\left[-\frac{(\lambda b)^2}{2}\right]$$

Using this, we see that the general term of the sum (6.1) will be

$$\left[\sum_{\rho=0}^{\infty} \frac{k_{2\rho}}{(2\rho)!} J_{2\rho}\right]^2 + \left[\sum_{\rho=0}^{\infty} \frac{k_{2\rho+1}}{(2\rho+1)!} K_{2\rho+1}\right]^2 = \exp\left[-(\lambda_m b)^2\right] [B_m^2 + C_m^2]$$

where

$$\left.\begin{aligned} B_m &= \sum_{\rho=0}^{\infty} (-1)^\rho \frac{k_{2\rho}}{(2\rho)!}\left(\frac{2\pi m}{p}\right)^{2\rho} \\ C_m &= \sum_{\rho=0}^{\infty} (-1)^{\rho+1} \frac{k_{2\rho+1}}{(2\rho+1)!}\left(\frac{2\pi m}{p}\right)^{2\rho+1} \end{aligned}\right\} \quad \ldots\ldots (6.5)$$

The k_ν have the values given by (6.3); if desired, the k_ν can also be expressed in terms of the semi-invariants or the moments of $\omega(z)$; the explicit expression for k_ν for any ν in terms of semi-invariants has been given by Frisch.[14]

The expression for D is finally

$$D^2 = \sum_{m=1}^{\infty} a_m \exp(-\kappa^2 m^2) \cdot (B_m^2 + C_m^2) \quad \ldots\ldots (6.6)$$

where $\kappa = 2\pi b/p$, a_m being given by (5.8), and B_m and C_m by (6.5).

In numerical work, the arbitrary constants a and b would usually be disposed of by putting a equal to the mean $M = \int_{-\infty}^{+\infty} \xi\omega(\xi)d\xi$, and b equal to the standard deviation $\sigma^2 = \int_{-\infty}^{+\infty} (\xi - M)^2\, \omega(\xi)\, d\xi$. If this be done, k_1 and k_2 will vanish in (6.5).

Since all the a_m in (6.6) are positive, and $a_1 + a_2 + \ldots = 1$, we have

$$D^2 \leq Q \quad \ldots\ldots (6.7)$$

where Q is the maximum of

$$Q_m = \exp(-\kappa^2 m^2) \cdot (B_m^2 + C_m^2)$$

for positive integral values of m. Hence we have the interesting

THEOREM V. The limit (6.7) for D^2 involves the period p, but otherwise is completely independent of the originator. In particular, the magnitude of the fluctuations of the originator does not influence the limit of the damping coefficient if the originator is periodic with period p.

If, for instance, the distribution function (i.e., the normalized distributor) is the normal, or Gaussian, error function, we have $B_m = 1$

[14] RAGNAR FRISCH. *Sur les semi-invariants et moments employés dans l'étude des distributions statistiques.* Skrifter utgitt av Det Norske Videnskaps Akademi, 1926: 23. Oslo.

and $C_m = 0$; hence $Q = \exp[-4(\pi\sigma/p)^2]$, where σ is the standard deviation of the normal error function which represents the normalized distributor. Consequently

$$D \leq \exp\left[-2\left(\frac{\pi\sigma}{p}\right)^2\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.8)$$

We therefore have

THEOREM VI. If the originator is periodic with period p, and the distributor is a normal error function with standard deviation σ, the damping coefficient is not greater than $\exp[-2(\pi\sigma/p)^2]$.

In other words, the ratio of σ and p has such a powerful influence on the magnitude of the fluctuations in the resultant that when the ratio σ/p becomes small a practically perfect equilibrium—in the sense of a small value for D—will be brought about, regardless of the magnitude of the fluctuations in the originator. The damping effect is such that, e.g., if $\sigma = p$, the value of D is less than 3.10^{-9}. In such cases, it will rarely ever be necessary for practical purposes to take into account the fluctuations of the originator—the resultant may, under such circumstances, be computed as if the originator were constant and equal to its actual mean value.

In view of Theorem V, the extremely low value of D found in the type of case just considered renders it reasonable to expect that a heavy damping will likewise take place when additional terms of the expansion (6.2) have to be taken into account. In each special case, the *exact* criteria have to be derived from (4.8) or (6.6) – (6.8); and the computed value of D has, of course, always to be viewed in relation to the nature of the particular problem under consideration.

It may be expected that a closer study of the more general cases, when the underlying trend is not constant, would reveal, in substance, the same characteristic features in the damping effect.

PALEOBOTANY—*A palm fruit from the Miocene of western Panama.*[1]

EDWARD W. BERRY, The Johns Hopkins University.

The Tertiary flora of Central America is very incompletely known so that considerable interest attaches to any additions that can be made to our knowledge of it. The palm fruit described below is such an addition. Unfortunately it has not been possible to assign this

[1] Received May 26, 1928.

fruit to an existing genus, but in extenuation it may be mentioned that not only are the recent palms of that region very incompletely known, but the majority are not represented by fruits in existing collections.

Twenty-two genera of palms are enumerated in Standley's Flora of the Canal Zone,[2] but all of these except the genera *Acanthorhiza* and *Geonoma* are quite different from the fossil, and there must be many species and even genera in Central America as a whole which remain undiscovered.

The present fossil is referred to the form genus *Palmocarpon*, and may be described as follows:

Palmocarpon geonomoides Berry, n. sp. (see fig. 1)

Fruit small, symmetrical, nut-like, nearly spherical—being 18.5 millimeters long and 17 millimeters in diameter. Lacking a raphe or any trace of a tripartite division, or of the basal pores of the cocoid palms. Fibres simple and flat, parallel with one another to the extreme base, not over 1.25 millimeters in maximum width, continuous from the base to within a millimeter or two of the apex, where their acute points are slightly raised as if free tipped in life; united by their edges to form a pseudo-shell.

x 2

Fig. 1.—*Palmocarpon geonomoides* Berry, n.sp., Miocene of western Panama.

The single specimen is a sandstone cast with a very slight film of carbonaceous matter in places. The general form is simulated in existing species of *Oenocarpus*, *Geonoma*, *Acanthorhiza*, etc., and the fibres suggest comparisons with existing species of *Rhopalostylis* and the outer coat of *Astrocaryum*. The fibres are, however, more regular than in *Astrocaryum*, especially toward the base, pores are lacking, and there is no trace of an inner fibrous layer which could hardly have disappeared during fossilization. Nor does it seem possible that the surface in life could have had several layers of imbricated lanceolate flat fibres without some trace of them having remained on the fossil. The fact that the specimen is fully inflated and perfectly round in transverse profile shows that it did not grow in a crowded inflorescence.

Because of the uncertainties of a positive generic identification and the lack of adequate recent comparative material it is referred to the form genus *Palmocarpon* instituted by Lesquereux in 1878 for palm fruits of uncertain generic affinities, the specific name being indicative

[2] STANDLEY, P. C. Contr. U. S. Nat. Herb. 27: 93–100. 1928.

of my opinion that it belongs to the tribe *Geonomeae*. The specimen comes from half a mile northwest of Zembrano, Province of Chiriqui, in western Panama, and was collected by James Terry Duce, and presented to the National Museum by T. D. A. Cockerell (U. S. N. M. no. 37193). It is labelled as having come from the Uscari formation, which on the Caribbean side of the Isthmus lies beneath the Gatun formation, but which has not been definitely recognized in Chiriqui Province. The age is undoubtedly Miocene.

Silicified palm wood was described by the writer[3] in 1918 from the Cucuracha formation (Oligocene) in the Gaillard Cut; and an Iriartea-like fruit was described[4] from the Gatun formation one and a half miles northeast of Gatun in 1921. These with the foregoing comprise all that is at present known of the Tertiary palms of Panama. No palms were represented in the small Tertiary florule described from Costa Rica in 1921,[5] but palm rays are not uncommon in collections from the Isthmus of Tehuantepec in southern Mexico.[6]

PROCEEDINGS OF THE ACADEMY AND AFFILIATED SOCIETIES

PHILOSOPHICAL SOCIETY

972D MEETING

The 972d meeting was held at the Cosmos Club March 17, 1928.

Program: F. L. MOHLER: *Recombination of atomic ions and electrons.* Spectra result from recombination of atomic ions and electrons. When a thermionic discharge tube designed to favor high concentration of ions was operated in caesium vapor with relatively high current and low voltage, continuous bands appeared at the limits of each series, in general agreement with theoretical predictions. Electrical measurements of discharge conditions have been made by the Langmuir probe wire method. The current voltage curves to the probes show that the electron velocities have a strictly random or temperature distribution with very low average speeds ranging between 0.2 and 0.3 volts. The ion concentration is of the order of 10^{13} per cc. On the basis of data on the intensity distribution beyond a limit and the electron velocity distribution one can compute the chance of recombination of an electron into the corresponding energy level as a function of the electron speed. This probability decreases very rapidly as the speed increases. From this can also be derived the probability of absorption of an atom in this state. The continuous emission bands beyond the 2P and 3D limits of caesium give similar absorption curves which drop rapidly from the limit to

[3] BERRY, EDWARD W. U. S. Nat. Mus. Bull. **103**: 24. 1918.
[4] BERRY, EDWARD W. Proc. U. S. Nat. Mus. **59**: 21–22. 1921.
[5] BERRY, EDWARD W. Idem. 169–185.
[6] BERRY, EDWARD W. Idem. **62** (Art. 19): *5, pl. 5, fig. 4.* 1923.

higher frequency. The form of curve is apparently quite different from the theoretical, which decreases approximately as the 4th power of the wave length. It resembles the curve for the 1S state derived by Foote and Mohler from photoionization measurements. (*Author's abstract.*)

R. M. LANGER: *Dispersion and quantum theory.* The interaction between radiation and matter can be expressed in terms of a property of the atom, namely the electric moment. The expressions for the rate of emission and absorption of light were given and discussed. The connection with the duration of emission or life of an atom in the excited state was mentioned. The relation between electric moment and absorption coefficients as well as refractive index was pointed out. Slides were shown of experimental results which proved that the positions of anomalous dispersion, absorption and emission were identical. This fact was decidedly in disagreement with the picture of the atom developed by Bohr and the discrepancy was one of the chief sources of dissatisfaction with this older form of quantum theory. The new quantum mechanics gives rise to a picture of the atom in which there is a movement of charge which matches in frequency the light given out, or absorbed. The positions of anomalous dispersion are then those to be expected from general electromagnetic theory and coincide as found in experiment with the absorption lines. We can therefore have more faith in the new model of the atom given by Schrödinger and it becomes interesting to calculate some of its properties. The distribution of charge in normal hydrogen was computed and a section of it was portrayed as the concentration of white dots on a black background. The atom looked like a spherical globule of charge with density decreasing gradually from the center outwards. The sphere with radius equal to the radius of the atom on the Bohr picture contained only half the charge. The considerable concentration outside this radius gives the impression of stray fields and fits in with our notions of the chemical notions of hydrogen. The excited state of hydrogen 2S was shown on a similar slide. It is also spherically symmetrical but the charge is not so concentrated, and is distributed as a diffuse shell concentric with the inner globule. In general S states of quantum number n would have n shells. The lithium atom is similar to 2S hydrogen, the sodium atom to 3S hydrogen, and so on. Normal helium was also shown and contrasted with hydrogen. The density at the center is very much higher but diminishes much more rapidly with radius so that practically all of the double electronic charge is within 10^{-8} cm. of the nucleus. This means there is very little stray field as we should expect from the fact that helium is an inert gas. The helium structure is characteristic of the outer shell of the elements of the second column of the periodic table, e.g., Hg and Cd. Another property which can be calculated with the help of the quantum mechanics is the atomic absorption coefficient. The total absorption of many molecules will involve an integration and not a simple multiplication by the number of molecules.

The mean life can also be computed and for the atom in the second quantum state has a value of about 1.58×10^{-9} seconds. (*Author's abstract.*)

973D MEETING

The 973d meeting was held at the Cosmos Club March 31, 1928.

Program: C. J. DAVISSON: *Reflection and diffraction of electrons by a crystal of nickel.* It has been found in recent experiments by the writer and

Dr. L. H. Germer that a stream of electrons is regularly reflected from a face of a nickel crystal. This phenomenon could be understood if electrons were waves, but is incomprehensible in terms of atoms and electrons and their interactions as we have pictured them. Reflection is not observed from a polycrystalline target, which suggests that if electrons are waves, their wave-lengths, like those of x-rays, must be comparable with the distances between atoms in solids. The reflection from the crystal is selective in speed of bombardment; intensity maxima occur at nearly equal intervals in this variable. This would be comprehensible if electrons were waves of wave-length inversely proportional to their speed. These results suggest by analogy with x-ray phenomena, that electrons will be regularly reflected also from sets of atom planes which do not lie parallel to the surface of the crystal. Beams of electrons of this type are found issuing from the crystal at critical speeds of bombardment, but their directions are not those of regular reflection from the principal atom planes. The difference between x-ray and electron diffraction in this respect is due apparently to a difference in the refractivity of nickel for the two kinds of radiation. It is shown that if this is the case the wave-length of the diffraction beam will nevertheless satisfy the plane grating formula with respect to the atomic plane grating lying parallel to the crystal face—files of atoms serving as the lines of a grating. This formula which does not involve the refractive index of the crystal is used to calculate electron wave-lengths, and all values so found agree, within the limits of accuracy of the measurements, with the corresponding values of h/mv—Planck's constant of action divided by the momentum of the incident electron. They agree, that is, with the wave-length associated with a freely moving particle in the undulatory mechanics introduced by L. de Broglie. The regularly reflected beams cannot be used for calculating wave-lengths. The Bragg formula cannot be used because the refractive index of the crystal is not unity, and the plane grating formula cannot be used because the beams are of order zero. Assuming h/mv to be the wave-length of the incident beam the data of regular reflection may, however, be used to calculate indices of refraction. When this is done it is found that the index of refraction of nickel for electrons decreases from about 1.15 to 1.01 as the bombarding potential is increased from 60 to 600 volts. (*Author's abstract.*)

H. L. CURTIS in an informal communication called attention to the close agreement between the recently measured velocity of light and value of the ohm, and the calculations of Rosa and Dorsey twenty-one years ago based upon their measurements of the ratio between the electromagnetic and electrostatic units of electricity.

974TH MEETING

The 974th meeting was held at the Cosmos Club April 14, 1928.

Program: L. W. KEPHART: *Plant hunting through East Africa.*

H. E. MERWIN, *Recording Secretary*

SCIENTIFIC NOTES AND NEWS

In connection with the Thirteenth Annual Meeting of the Optical Society of America an exhibition of optical instruments, apparatus and products will be held under the joint auspices of the Bureau of Standards and the Optical Society, from October 31 to November 3.

Dr. A. S. HITCHCOCK has returned from Newfoundland and Labrador, where he has been studying and collecting grasses. In Newfoundland he collected at Port-aux-Basques, St. Georges, Corner Brook, Little Harbor, Grand Falls, and St. Johns, and in Labrador at Battle Harbor and Cartwright.

Dr. C. G. ABBOT, Secretary of the Smithsonian Institution, returned to Washington on September 23 after a successful expedition to Mount Wilson, California. Bolometric observations of the infra-red solar spectra and radiometric observations of the spectra of many stars formed his principal work. The results, which have not yet been fully worked up, appear to be very gratifying.

Mr. M. W. STIRLING, Chief of the Bureau of American Ethnology, attended the meeting in New York City of the XXIII Session of the International Congress of Americanists, which convened on Monday, September 17. Mr. Stirling was appointed by the State Department to represent the United States Government at the Congress. He returned to Washington September 24.